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SOME ANNULAR DISC INCLUSION PROBLEMS IN ELASTICITY

[redacted]

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[redacted]

Abstract—This paper examines the problems related to the displacement and rotation of a rigid annular disc inclusion which is embedded in bonded contact with an isotropic elastic infinite space. The analysis of the [redacted]

1. INTRODUCTION

The class of problems related to the behaviour of flexible or rigid disc shaped inclusions [redacted] studies by Collins[1] and Keer[2] examine the problems of a rigid penny shaped inclusion embedded in bonded contact with an isotropic elastic solid. These studies were subsequently [redacted] inclusions and inhomogeneities embedded in elastic media are given by Mura[13], Willis[14] and Walpole[15].

This paper examines a series of axisymmetric and asymmetric problems related to an [redacted] translation of the inclusion in the x-direction. The rotationally symmetric deformations are induced by the torsion of the annular disc inclusion about the z-axis. By virtue of the [redacted]

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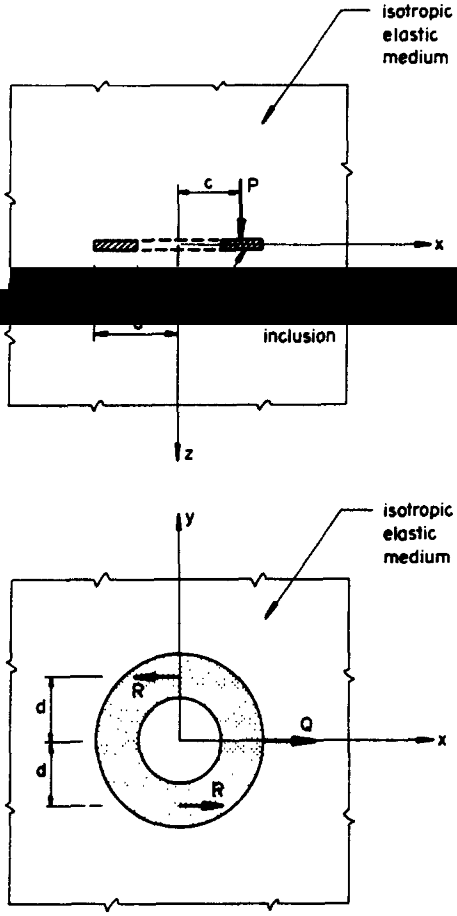
[redacted]



[redacted]

Fig. 1. Geometry of the annular disc inclusion and the resultant forces.

elastic materials. In geomechanical applications the rigid disc shaped inclusion represents the [redacted]

strengthen non-metallic or metallic matrices or increase the overall stiffness of a composite [redacted]

In connection with the solution of the axisymmetric and asymmetric problems related to the embedded annular inclusion it is convenient to employ a formulation based on the strain potential approach of Love[21] and its extension to asymmetric problems proposed by [redacted]

Somigliana–Galerkin stress function[23, 24]. Proofs of the completeness of these represen- [redacted] function $\bar{\Psi}(r, \theta, z)$, i.e.:

$$\nabla^4\Phi(r, \theta, z) = 0; \nabla^2\Psi(r, \theta, z) = 0 \tag{1}$$

where

$$\nabla^4 = \nabla^2\nabla^2$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

[redacted]

We have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\partial z} + \frac{2}{r}\frac{\partial\Psi}{\partial\theta} \tag{3a}$$

$$2Gu_\theta = -\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z} - 2\frac{\partial\Psi}{\partial r} \tag{3b}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{3c}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively. Similarly, the components of the stress tensor $\boldsymbol{\sigma}$ are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\Phi + \frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\Psi \tag{4a}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi - \frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\Psi \tag{4b}$$

[redacted]

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi - \frac{\partial^2\Psi}{\partial r\partial z} \tag{4d}$$

[redacted]

[redacted]

It may be noted that for axial symmetry $\Phi = \Phi(r, z)$ and $\Psi = 0$; thus the results (3) and (4) for

[redacted]

[redacted]

stresses, in the infinite space, about the plane $z = 0$. We may therefore restrict the analysis to a single halfspace region in which the plane $z = 0^+$ is subjected to appropriate mixed boundary

[redacted]

(i) *For the rigid body translation in the z-direction*

[redacted]

$$u_r(r, 0^+) = 0; r \geq 0 \tag{5a}$$

$$u_z(r, 0^+) = \delta; b \leq r \leq a \tag{5b}$$

[redacted]

$$\sigma_{zz}(r, 0^+) = 0; 0 < r < b. \tag{5d}$$

(ii) *For the rigid body rotation about the y-axis*

$$u_r(r, \theta, 0^+) = 0; r \geq 0 \tag{6a}$$

$$u_\theta(r, \theta, 0^+) = 0; r \geq 0 \tag{6b}$$

$$u_z(r, \theta, 0^+) = \Omega r \cos\theta; b \leq r \leq a \tag{6c}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0;\ a < r < \infty \tag{6d}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; 0 < r < b. \tag{6e}$$

(iii) *For the rigid body rotation about the z-axis*

$$u_\theta(r, \theta, 0^+) = \omega r; b \leq r \leq a \tag{7a}$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; a < r < \infty \tag{7b}$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; 0 < r < b. \tag{7c}$$

(iv) *For the rigid body translation along the x-direction*

$$u_z(r, \theta, 0^+) = 0; r \geq 0 \tag{8a}$$

$$u_r(r, \theta, 0^+) = \delta \cos\theta; b \leq r \leq a \tag{8b}$$

$$u_\theta(r, \theta, 0^+) = -\delta \sin\theta; b \leq r \leq a \tag{8c}$$

$$\sigma_{rz} \sin\theta + \sigma_{\theta z} \cos\theta = 0; r \geq 0 \tag{8d}$$

$$\sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta = 0; a < r < \infty \tag{8e}$$

$$\sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta = 0; 0 < r < b. \tag{8f}$$

The boundary conditions (8d), (8e) and (8f) relate to the traction vectors which act on the plane $z = 0^+$ along the y and x directions, respectively.

[redacted]

[illegible]

solutions for Φ and Ψ take the following forms.

(i) *For the rigid body translation in the z-direction*

$$\Phi(r, z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{9a}$$

$$\Psi(r, z) = 0. \tag{9b}$$

(ii) *For the rigid body rotation about the y-axis*

$$\Phi(r, \theta, z) = \left\{\int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \cos\theta \tag{10a}$$

$$\Psi(r, \theta, z) = \left\{\int_0^\infty \xi C(\xi)\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \sin\theta. \tag{10b}$$

(iii) *For the rigid body rotation about the z-axis*

$$\Phi(r, \theta, z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_1(\xi r)\, d\xi \tag{11a}$$

$$\Psi(r, \theta, z) = 0. \tag{11b}$$

(iv) *For the rigid body translation along the x-direction*

$$\Phi(r, \theta, z) = \left\{\int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \cos\theta \tag{12a}$$

$$\Psi(r, \theta, z) = \left\{\int_0^\infty \xi C(\xi)\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \sin\theta. \tag{12b}$$

For convenience, we define the nth order Hankel operator as follows:

$$H_n[f(\xi); r] = \int_0^\infty \xi f(\xi) J_n(\xi r)\, d\xi. \tag{13}$$

[illegible] the displacement and stress components given by (3a–c) and (4a–f) it can be shown that the mixed boundary conditions (5)–(8) reduce to sets of triple integral equations for an unknown function $R_n(\xi)(n = 1,2,3,4)$.

(i) For the rigid body displacement of the annular disc inclusion in the z-direction we have

$$H_0[R_1(\xi); r] = 0;\ 0 < r < b, \tag{14a}$$

$$H_0[\xi^{-1}R_1(\xi); r] = -\frac{2\delta(1-\nu)}{(3-4\nu)};\ b \le r \le a \tag{14b}$$

$$H_0[R_1(\xi); r] = 0;\ a < r < \infty. \tag{14c}$$

[redacted]

$$H_1[\xi^{-1}R_2(\xi); r] = 0; 0 < r < b \tag{15a}$$

$$H_1[R_2(\xi); r] = -\frac{2\Omega r(1-\nu)}{(3-4\nu)}; b \le r \le a \tag{15b}$$

$$H_1[\xi^{-1}R_2(\xi); r] = 0; a < r < \infty. \tag{15c}$$

[redacted]

$$H_1[R_3(\xi); r] = 0; 0 < r < b \tag{16a}$$

$$H_1[\xi^{-1}R_3(\xi); r] = \omega r; b \le r \le a \tag{16b}$$

[redacted]

(iv) For the lateral translation of the annular disc inclusion along the x-direction we have

$$H_1[R_4(\xi); r] = 0; 0 < r < b \tag{17a}$$

$$H_1[\xi^{-1}R_4(\xi); r] = -\frac{4\Delta(1-\nu)}{(7-8\nu)}; b \le r \le a \tag{17b}$$

$$H_1[R_4(\xi); r] = 0; a < r < \infty. \tag{17c}$$

The sets of triple integral equations defined by (14)–(17) can be solved by employing a variety [redacted] the method of solution proposed by Williams[20]. In its general form, the triple system can be written as

$$H_n[R(\xi); r] = 0; 0 < r < b \tag{18}$$

$$H_n[\xi^{-1}R(\xi); r] = f(r); b \le r \le a \tag{19}$$

$$H_n[R(\xi); r] = 0; a < r < \infty. \tag{20}$$

[redacted]

$$H_n[R(\xi); r] = g(r); b < r < a. \tag{21}$$

From the Hankel inversion theorem we have

$$R(\xi) = \int_a^b rg(r)J_n(\xi r)\,dr. \tag{22}$$

Using this result in (19) we obtain

$$\int_a^b ug(u)K_0(u, r)\,du = f(r); b \le r \le a \tag{23}$$

where

$$K_0(u, r) = u\int_0^\infty J_n(\xi r)J_n(\xi u)\,d\xi. \tag{24}$$

We define the functions $g_1(u)$ and $g_2(u)$ such that

$$g_1(u)+g_2(u)=\begin{cases}0 & ;0\le r<b\\ g(u) & ;b\le r\le a\\ 0 & ;a<r<\infty\end{cases} \tag{25}$$

and assume that $f(r)$ admits expansions of the form

[redacted]

$$f_2(r)=\sum_{n=-\infty} a_n r^n;\ b<r<\infty. \tag{27}$$

From the representations (24)–(27) it follows that the integral equation (23) reduces to two integral equations

$$\int_0^\infty uK_0(u,r)g_1(u)\,\mathrm{d}u=f_1(r);0<r<a \tag{28}$$

$$\int_0^\infty uK_0(u,r)g_2(u)\,\mathrm{d}u=f_2(r);b<r<\infty. \tag{29}$$

[redacted]

$$\int_0^\infty tK_0(r,t)g(t)\,\mathrm{d}t=4r^{-n}\int_0^r\frac{s^{2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_s^\infty\frac{t^{1-n}g(t)\,\mathrm{d}t}{(t^2-s^2)^{1/2}};0<r<\infty \tag{30}$$

$$\int_0^\infty tK_0(r,t)g(t)\,\mathrm{d}t=4r^{n}\int_r^\infty\frac{s^{-2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_0^s\frac{t^{1+n}g(t)\,\mathrm{d}t}{(s^2-t^2)^{1/2}};0<r<\infty. \tag{31}$$

Using these results, the integral equations (28) and (29) can be expressed in the form

$$4r^{-n}\int_s^r\frac{s^{2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_s^\infty\frac{t^{1-n}g_1(t)\,\mathrm{d}t}{(t^2-s^2)^{1/2}}=f_1(r);0<r<a \tag{32}$$

$$4r^{n}\int_r^\infty\frac{s^{-2n}\,\mathrm{d}s}{(s^2-r^2)^{1/2}}\int_0^s\frac{t^{1+n}g_2(t)\,\mathrm{d}t}{(s^2-t^2)^{1/2}}=f_2(r);b<r<\infty. \tag{33}$$

The next step is to define unknown functions $S_i(r)$, $T_i(r)$ and $C_i(r)$ $(i=1,2)$ such that

[redacted]

$$r^{-n}\int_0^r\frac{t^{1+n}g_2(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}}=\begin{cases}-T_2(r);0<r<b\\ S_2(r);b<r<\infty\end{cases} \tag{35}$$

$$4r^{-n}\int_0^r\frac{s^nC_1(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}}=f_1(r);0<r<a \tag{36}$$

$$4r^{n}\int_r^\infty\frac{s^{-n}C_2(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}}=f_2(r);b<r<\infty \tag{37}$$

$$S_i(r)=C_i(r). \tag{38}$$

The four integral equations (35)–(38) can be inverted to give

[redacted]

$$g_2(t) = \frac{2}{\pi} t^{-n-1} \frac{d}{dt}\left[-\int_0^b \frac{u^{n+1} T_2(u)\, du}{(t^2 - u^2)^{1/2}} + \int_b^t \frac{u^{n+1} S_2(u)\, du}{(t^2 - u^2)^{1/2}} \right] \tag{40}$$

$$C_1(r) = \frac{1}{2\pi r^n} \frac{d}{dr} \int_0^r \frac{u^{n+1} f_1(u)\, du}{(r^2 - u^2)^{1/2}};\ 0 < r < a \tag{41}$$

$$C_2(r) = -\frac{r^n}{2\pi} \frac{d}{dr} \int_r^\infty \frac{u^{1-n} f_2(u)\, du}{(u^2 - r^2)^{1/2}};\ b < r < \infty. \tag{42}$$

[redacted]

$$T_1(r) = l_1(r) + \frac{n!}{r^n \sqrt{(\pi)}\Gamma(n + (3/2))} \int_0^b \frac{u^{n+1} T_2(u)\, {}_2F_1((1/2), n; n + (3/2); (u^2/r^2))\, du}{(r^2 - u^2)} \quad ; a < r < \infty \tag{43}$$

[redacted]

where ${}_2F_1$ is a hypergeometric function and $l_1(r)$ and $l_2(r)$ are given by

$$l_1(r) = -\frac{2}{\pi r^n} \int_0^r \frac{t^{2n}\, dt}{(r^2 - t^2)^{1/2}} \frac{d}{dt} \int_t^a \frac{u^{1-n} S_1(u)\, du}{(u^2 - t^2)^{1/2}} \tag{45}$$

[redacted]

The integral equations (43) and (44) can be solved, by using iterative techniques, to yield expressions for $T_i(r)$; these in turn can be used in (39) and (40) to generate the expressions for $g_i(t)$. Specific results derived from the method are outlined below.

(i) For example, for the rigid body displacement of the disc inclusion in the axial direction we have

$$n = 0;\ f(r) = A = \text{const.};\ f_1(r) = A;\ f_2(r) = 0. \tag{47}$$

From (38), (41) and (42) we have

$$C_1 = S_1 = \frac{A}{2\pi};\ C_2 = S_2 = 0. \tag{48}$$

Making use of (43)–(46) we find that

$$l_1(br) = \frac{1}{\pi^2}\left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + 0(\lambda^9)\right]$$

$$l_2(ar) = 0 \tag{49}$$

$\lambda = b/a$ and $0(\lambda^n)$ is the Landau symbol.

By iteration we obtain from (43) and (44) the following expressions for $T_i(r)$:

[redacted]

$$T_1(ar) = \frac{2\lambda}{\pi^2}\left[\frac{1}{r^2}\left(\frac{\lambda}{3} + \frac{\lambda}{15} + \frac{4\lambda}{27\pi^2} + \frac{\lambda}{35} + \frac{92\lambda}{675\pi^2}\right) + \frac{1}{r^4}\left(\frac{\lambda^3}{5} + \frac{\lambda^5}{21} + \frac{4\lambda^6}{45\pi^2}\right) + \frac{\lambda^5}{7\pi^6} + 0(\lambda^7)\right];\ 1 < r < \infty \tag{50}$$

$$T_2(br) = \frac{1}{\pi^2}\left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + \frac{4}{\pi^2}\left\{\left(\frac{\lambda^4}{9} + \frac{14\lambda^6}{225} + \frac{4\lambda^7}{81\pi^2} + \frac{29\lambda^8}{735}\right) r + \left(\frac{\lambda^6}{15} + \frac{22\lambda^8}{525}\right) r^3 + \frac{\lambda^2 r^5}{21}\right\} + 0(\lambda^9)\right]; 0 < r < 1. \quad (51)$$

These results can be used to develop the relevant expression for $g(t)(= g_1(t) + g_2(t))$.

[redacted]

where B is a constant

$$C_1(r) = S_1(r) = 2Br;\ C_2(r) = S_2(r) = 0.$$

The corresponding expressions for $T_1(ar)$ and $T_2(br)$ take the forms:

$$T_1(ar) = \frac{32Ba\lambda^5}{45\pi^2}\left[\frac{1}{r^3}\left(1 + \frac{2\lambda^2}{7}\right) + \frac{6\lambda^2}{7r^5} + 0(\lambda^4)\right]; 1 < r < \infty \quad (52)$$

[redacted]

Similar results can be derived for the problems which relate to rotation of the disc inclusion [redacted]

[redacted]

applications.

(i) *Rigid body translation in the z-direction*

[redacted]

y-axis. Considering the axisymmetric problem we have

$$P = 2\pi \int_b^a r[\sigma_{zz}(r, 0^-) - \sigma_{zz}(r, 0^+)]\,dr \quad (54)$$

where $\sigma_{zz}(r, 0^+)$ and $\sigma_{zz}(r, 0^-)$ refer to the normal interface stresses which act on the faces of the [redacted]

Considering (47) we can set $A = -2\delta(1-\nu)/(3-4\nu)$; consequently (55) yields

$$P = \frac{64(1-\nu)Ga\delta}{(3-4\nu)}\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + 0(\lambda^9)\right]. \quad (56)$$

We note that as $\lambda \to 0$, (56) reduces to the classical result for the solid disc inclusion derived by Collins[1], Kanwal and Sharma[19] and Selvadurai[18].

(ii) *Rigid body rotation about the* y-*axis*

The resultant moment $M_0 = Pc$ is given by

$$M_0 = \pi \int_b^a r^2[\sigma_{zz}(r, \theta, 0^-) - \sigma_{zz}(r, \theta, 0^+)]\, dr. \tag{57}$$

Again $\sigma_{zz}(r, \theta, 0^+) = -\sigma_{zz}(r, \theta, 0^-)$ and

$$M_0 = -4\pi G \int_b^a r^2 g(r)\, dr. \tag{58}$$

[redacted]

Again as $\lambda \to 0$, (59) reduces to the result for the solid inclusion given by Selvadurai[7]

[redacted]

(iii) *Rigid body rotation about the* z-*axis*

The forces R act in the plane of the disc inclusion. These forces are equivalent to a resultant

[redacted]

$$T = 2\pi \int_b r^2[\sigma_{\theta z}(r, 0^-) + \sigma_{\theta z}(r, 0^+)]\, dr \tag{60}$$

Using the results derived in the previous sections it can be shown that

$$T = \frac{32Ga^3\omega}{3}\left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + 0(\lambda^9)\right]. \tag{61}$$

The result (61) is in agreement with analogous results derived by Collins[28] for the Reissner–Sagoci problem for an annular punch.

(iv) *Rigid body translation along the* x-*axis*

The application of the force Q causes a rigid body translation (Δ) of the annular disc inclusion along the x-direction.

$$Q = \int_b^a \int_0^{2\pi} [T_x(r, \theta, 0^+) + T_x(r, \theta, 0^-)] r\, dr\, d\theta. \tag{62}$$

The load–displacement relationship takes the form

$$Q = \frac{64(1-\nu)Ga\Delta}{(7-8\nu)}\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + 0(\lambda^9)\right]. \tag{63}$$

As $\lambda \to 0$, (63) reduces to the results given by Keer[2], Kassir and Sih[3] and Selvadurai[9] for the lateral translation of the embedded solid disc inclusion.

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APPENDIX A

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{J_{n-1/2}(ps)s^{n+(1/2)}\,ds}{(r^2 - s^2)^{1/2}} \tag{A1}$$

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} r^n \int_r^\infty \frac{J_{n+(1/2)}(ps)s^{-n+(1/2)}\,ds}{(s^2 - r^2)^{1/2}} \tag{A2}$$

$$\int_0^\infty pJ_{n+(1/2)}(ps)J_{n+(1/2)}(pt)\,dp = \frac{\delta^*(s-t)}{(st)^{1/2}} \tag{A3}$$

where δ^* is the Dirac delta function.