Int. J. Engng Sci. Vol. 25, No. 8, pp. 1049–1057, 1987 0020-7225/87 $3.00 + 0.00
[redacted]

[redacted] PENNY-SHAPED CRACK

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[redacted] factors at the boundary of the penny-shaped flaw and at the perimeter of the externally cracked [redacted]

1. INTRODUCTION

The stress analysis of a penny-shaped crack located in an isotropic elastic solid is a classical problem in linear elastostatics. It is also a problem of fundamental interest to the study of initiation and propagation of fracture in brittle solids. The classical studies of the penny-shaped flaw problem are given by Sneddon [1, 2] and Sack [3] and detailed accounts of further developments in the stress analysis of penny-shaped defects located in elastic media are given by Sneddon and Lowengrub [4], Kassir and Sih [5] and Cherepanov [6]. These latter references contain complete accounts of problems in which the penny-shaped crack is subjected to arbitrary surface tractions. The class of problems in which the surfaces of the penny-shaped flaw is subjected to displacement dependent traction boundary conditions has been investigated by Selvadurai [7, 8] in connection with the analysis of flaw-bridging in unidirectional fibre reinforced materials. Recently Selvadurai and Singh [9] have [redacted] elastic solid which is weakened by an external crack situated in the plane of the penny-shaped crack (Fig. 1). The crack is subjected to uniform internal pressure of intensity p_0. The analysis of the axisymmetric mixed boundary value problem is achieved by employing a Hankel transform development of the governing field equations. The mixed boundary [redacted] fashion. The analysis of the problem concentrates on the evaluation of the stress intensity factors at the boundary of the penny-shaped crack and at the boundary of the externally cracked region. These stress intensity factors are evaluated in power series form in terms of a non-dimensional parameter which involves the ratio of the radius of the penny-shaped crack to the radius of the externally cracked region.

It may be noted that due to the existence of the infinite crack and in view of the uniform loading, the principle of superposition does not hold for the problem examined. [redacted] problems [11, 12]. In the absence of body forces the solution of the displacement equations

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$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left\{\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right\} \tag{6}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \tag{7}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\}. \tag{8}$$

[redacted]

by a penny-shaped flaw of radius "a" and bounded externally by an external circular crack [redacted] problem exhibits a state of symmetry about the plane $z = 0$, we can restrict our attention to a single halfspace occupying the region $z \geqslant 0$ and denote by $z = 0^+$ the plane of symmetry associated with that region. The mixed boundary conditions relevant to the crack problem are as follows:

$$\sigma_{zz}(r, 0^+) = -f(r) = -p_0; \qquad 0 < r < a \tag{9}$$

$$\sigma_{zz}(r, 0^+) = 0; \qquad r > b \tag{10}$$

$$u_z(r, 0^+) = 0; \qquad a \leqslant r \leqslant b \tag{11}$$

$$\sigma_{rz}(r, 0^+) = 0; \qquad r \geqslant 0. \tag{12}$$

In order to examine the mixed boundary value problem defined by (9)–(12) it is convenient [redacted] $\Phi(r, z)$ can be chosen such that the stresses and displacements derived from $\Phi(r, z)$ reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$. The relevant solution is (see e.g. Sneddon [13])

$$\Phi(r, z) = \int_0^\infty [A_1(\xi) + zA_2(\xi)]\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{13}$$

[redacted] integral equations for a single unknown function $A(\xi)$ [the function $A_1(\xi)$ and $A_2(\xi)$ can be expressed in terms of $A(\xi)$]; we have

$$H_0[\xi^{-1}A(\xi); r] = f(r); \qquad 0 < r < a \tag{14}$$

$$H_0[\xi^{-2}A(\xi); r] = 0; \qquad a \leqslant r \leqslant b \tag{15}$$

$$H_0[\xi^{-1}A(\xi); r] = 0; \qquad b < r < \infty. \tag{16}$$

where

$$H_0[F(\xi); r] = \int_0^\infty \xi F(\xi) J_0(\xi r)\, \mathrm{d}\xi. \tag{17}$$

The system of triple integral equations defined by (14)–(16) can be solved by employing the procedures described by Cooke [14]. Complete accounts of the techniques that may be employed in the solution of systems of triple integral equations are given by Williams [15], Tranter [16], Sneddon [17] and Kanwal [18]. In the ensuing we shall present a brief summary of the analytical procedure which focusses on the evaluation of an asymptotic series solution in terms of a small parameter.

We assume that

$$H_0[\xi^{-2}A(\xi); r] = \begin{cases} f_1(r); & 0 < r < a \quad (18) \\ f_2(r); & b < r < \infty. \quad (19) \end{cases}$$

Employing the results given by Cooke [14] it can be shown that

$$f_1(r) = p_1(r) + \frac{2}{\pi} \int_b^\infty \frac{t f_2(t)(a^2 - r^2)^{1/2}\,\mathrm{d}t}{(t^2 - a^2)^{1/2}[t^2 - r^2]}; \qquad 0 < r < a \tag{20}$$

and

[illegible] $\pi \int_0 (b^2 - t^2)^{1/2}[r^2 - t^2]$, [illegible] (21)

where

$$p_1(r) = \frac{2}{\pi} \int_r^a \left\{ \int_0^s \frac{t f(t)\,\mathrm{d}t}{(s^2 - t^2)^{1/2}} \right\} \frac{\mathrm{d}s}{(s^2 - r^2)^{1/2}}. \tag{22}$$

Introduce functions $F_1(r)$ and $F_2(r)$ such that

$$F_1(r) = \frac{\mathrm{d}}{\mathrm{d}r} \int_r^a \frac{t f_1(t)\,\mathrm{d}t}{(t^2 - r^2)^{1/2}}; \qquad 0 < r < a \tag{23}$$

and

$$F_2(r) = \frac{\mathrm{d}}{\mathrm{d}r} \int_b^r \frac{t f_2(t)\,\mathrm{d}t}{(r^2 - t^2)^{1/2}}; \qquad b < r < \infty. \tag{24}$$

The solutions of the above Abel integral equations take the forms

$$f_1(t) = -\frac{2}{\pi} \int_t^a \frac{F_1(s)\,\mathrm{d}s}{(s^2 - t^2)^{1/2}}; \qquad 0 < t < a \tag{25}$$

$$f_2(t) = \frac{2}{\pi} \int_b^t \frac{F_2(s)\,\mathrm{d}s}{(t^2 - s^2)^{1/2}}; \qquad b < t < \infty. \tag{26}$$

By making use of (20) and (25) we have

$$\int_r^a \frac{F_1(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}} = -\frac{\pi}{2}p_1(r) - \int_b^\infty \frac{t(a^2-r^2)^{1/2}f_2(t)\,\mathrm{d}t}{(t^2-a^2)^{1/2}(t^2-r^2)}; \qquad 0<r<a. \tag{27}$$

[redacted]

$$F_1(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{rp_1(r)\,\mathrm{d}r}{(r^2-s^2)^{1/2}} + \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{r(a^2-r^2)^{1/2}}{(r^2-s^2)^{1/2}} \times \left\{\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2-a^2)^{1/2}(t^2-r^2)}\right\}\mathrm{d}r; \qquad 0<s<a. \tag{28}$$

The second integral of (28) can be reduced to the form (see Appendix A)

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2-a^2)^{1/2}}\int_s^a \frac{r(a^2-r^2)^{1/2}\,\mathrm{d}r}{(r^2-s^2)^{1/2}(t^2-r^2)} = s\int_b^\infty \frac{F_2(u)\,\mathrm{d}u}{(u^2-s^2)}. \tag{29}$$

Consequently (28) gives

$$F_1(s) + \frac{2s}{\pi}\int_b^\infty \frac{F_2(u)\,\mathrm{d}u}{(u^2-s^2)} = \frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{rp_1(r)\,\mathrm{d}r}{(r^2-s^2)^{1/2}}; \qquad 0<s<a. \tag{30}$$

[redacted]

In a similar fashion (21) can be reduced to the form

$$F_2(s) + \frac{2}{\pi}\int_0^a \frac{uF_1(u)\,\mathrm{d}u}{(s^2-u^2)} = 0; \qquad b<s<\infty. \tag{31}$$

The eqns (30) and (31) are a pair of simultaneous Fredholm integral equations which can [redacted] particular instance when $f(r) = p_0$, the eqn (30) can be written

[redacted]

where $c = a/b$; $s_1 = s/a$ and $u_1 = u/b$. Similarly by introducing the substitutions $u_1 = u/a$ and $s_1 = s/b$, the eqn (31) can be written as

$$F_2(bs_1) + \frac{2c^2}{\pi}\int_0^1 \frac{u_1F_1(au_1)\,\mathrm{d}u_1}{(s_1^2-c^2u_1^2)} = 0; \qquad 1<s_1<\infty. \tag{33}$$

[redacted]

$$(u_1^2 - s_1^2c^2)^{-1} = \frac{1}{u_1^2} + \frac{s_1^2c^2}{u_1^4} + \frac{s_1^4c^4}{u_1^6} + \frac{s_1^6c^6}{u_1^8} + \frac{s_1^8c^8}{u_1^{10}} + O(c^{10}) \tag{34}$$

$$(s_1^2 - u_1^2c^2)^{-1} = \frac{1}{s_1^2} + \frac{u_1^2c^2}{s_1^4} + \frac{u_1^4c^4}{s_1^6} + \frac{u_1^6c^6}{s_1^6} + \frac{u_1^8c^8}{s_1^8} + O(c^{10}) \tag{35}$$

[redacted]

$$F_1(as_1) = \sum_{i=0}^{8} c^i m_i(s_1) \tag{36}$$

$$F_2(bs_1) = \sum_{i=0}^{8} c^i n_i(s_1). \tag{37}$$

By substituting (34)–(37) in eqns (32) and (33) and comparing like terms in c^i it is possible to determine the functions $m_i(s_i)$ and $n_i(s_1)$. We have

$$F_1(as_1) = ap_0\left[-s_1 - \frac{4s_1}{9\pi^2}c^3 - \frac{4c^5}{5\pi^2}\left(\frac{s_1}{5}+\frac{s_1^3}{3}\right) - \frac{16c^6 s_1}{81\pi^4} + \frac{4s_1c^7}{\pi^2}\left(\frac{1}{49}+\frac{s_1^2}{35}+\frac{s_1^4}{21}\right) - \frac{16c^8}{\pi^4}\left(\frac{s_1}{75}+\frac{s_1^3}{135}\right) + O(c^9)\right]; \quad 0 < s_1 < 1 \tag{38}$$

[redacted]

$$+\frac{16c^7}{\pi^3 s_1^2}\left(\frac{1}{75}+\frac{1}{90s_1^2}\right) + \frac{2c^8}{9\pi s_1^2}\left(\frac{16}{27\pi^4}+\frac{1}{s_1^6}\right) + O(c^9)\bigg]; \quad 1 < s_1 < \infty. \tag{39}$$

This formally completes the analysis of the problem and the function $A(\xi)$ can be expressed in the form

$$A(\xi) = \xi^2\left[-\int_0^a F_1(s)\,\mathrm{d}s\int_0^s \frac{rJ_0(\xi r)\,\mathrm{d}r}{(s^2-r^2)^{1/2}} + \int_b^\infty F_2(s)\,\mathrm{d}s\int_s^\infty \frac{rJ_0(\xi r)\,\mathrm{d}\xi}{(r^2-s^2)^{1/2}}\right]. \tag{40}$$

4. THE STRESS INTENSITY FACTOR

In the ensuing we shall examine the stress intensity factors associated with the boundary of the penny-shaped crack and the circular boundary of the externally cracked region. Considering (40) and the result

$$\sigma_{zz}(r,0) = -H_0[\xi^{-1}A(\xi); r] \tag{41}$$

it can be shown that

$$\sigma_{zz}(r,0) = \left[\frac{-F'_1(a)}{(r^2-a^2)^{1/2}} + \int_0^a \frac{F_1(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}} + \frac{F'_2(b)}{(b^2-r^2)^{1/2}} + \int_b^\infty \frac{F'_2(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}}\right] \tag{42}$$

where $F'_1(s)$ and $F'_2(s)$ denote derivatives of the respective functions.

The stress intensity factors at the boundaries $r = a$ and $r = b$ are defined by

$$K_a = \lim_{r\to a^+}[2(r-a)]^{1/2}\sigma_{zz}(r,0) \tag{43}$$

and

$$K_b = \lim_{r\to b^-}[2(b-r)]^{1/2}\sigma_{zz}(r,0) \tag{44}$$

respectively. From (42), (43) and (44) it follows that

$$K_a = -\frac{F_1(a)}{\sqrt{a}} \tag{45}$$

[redacted]

Using eqns (38) and (39) in (45) and (46) we obtain the following expressions for the stress intensity factors:

$K_a = p_0\sqrt{a}\Big[1 +$ [redacted] $+ O(c^9)\Big]$ (47)

[redacted]

$$K_b = p_0\sqrt{a}\sqrt{c}\left[\frac{2c^2}{3\pi} + \frac{2c^4}{5\pi} + \frac{8c^5}{27\pi^3} + \frac{2c^6}{7\pi} + \frac{88c^7}{225\pi^3} + \frac{2c^8}{9\pi}\left(1 + \frac{16}{27\pi^4}\right) + O(c^9)\right] \tag{48}$$

where $c = a/b$.

The expression for the displacement u_z can be written in the form

$$u_z(r,0) = \begin{cases} \dfrac{-4}{\pi}\dfrac{(1-\nu^2)}{E}\displaystyle\int_r^a \frac{F_1(s)\,ds}{(s^2-r^2)^{1/2}}; & 0 \leqslant r \leqslant a \quad (49a) \\[2ex] \dfrac{4}{\pi}\dfrac{(1-\nu^2)}{E}\displaystyle\int_b^r \frac{F_2(s)\,ds}{(r^2-s^2)^{1/2}}; & b \leqslant r \leqslant \infty \quad (49b) \end{cases}$$

The work done in opening the penny-shaped crack is given by

$$W = 2\pi p_0 \int_0^a r u_z(r,0)\,dr \tag{50}$$

From (49a) and (50) we obtain

[redacted]

5. CONCLUSIONS

The computed stress intensity factors (47) and (48) indicate that when the penny-shaped crack is subjected to uniform internal pressure the stress intensity factor at the boundary [redacted] of the weakened zone ($r = b$). Consequently brittle-elastic type of fracture will be initiated [redacted]

factor at the boundary of the penny-shaped crack remains at its classical value for

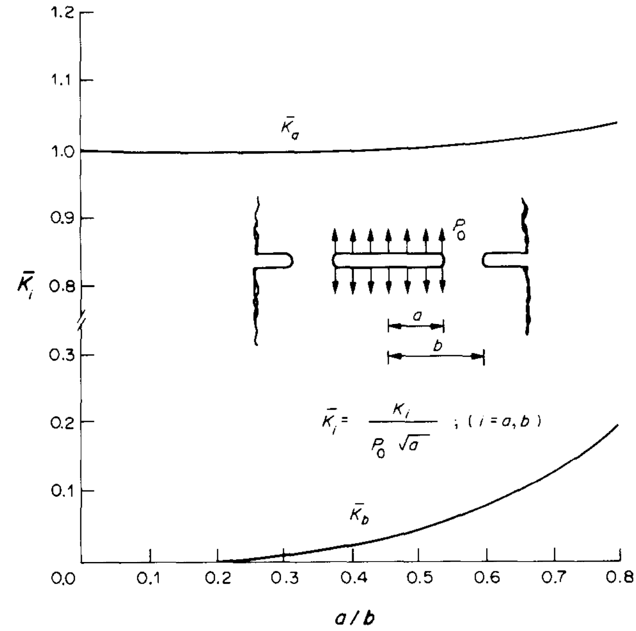

Fig. 2

[redacted] crack does not result in the development of appreciable stress intensity factors at the external crack region. Owing to the nature of the asymptotic analysis the accuracy of the solution presented is expected to be satisfactory for small values of $c\varepsilon(0, 0.7)$.

Acknowledgements—The work described in this paper was supported in part by a Natural Sciences and Engineering Research Council of Canada Grant A3866 awarded to the first author.

[redacted]

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(*Received* 1 *November* 1986)

APPENDIX A

The integral

[redacted]

$$= \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2 - a^2)^{1/2}}\int_s^a \frac{r(a^2 - r^2)^{1/2}\mathrm{d}r}{(r^2 - s^2)^{1/2}(t^2 - r^2)}. \tag{A2}$$

[redacted]

$$\int_s^a \frac{r(a^2 - r^2)\,\mathrm{d}r}{(r^2 - s^2)^{1/2}(t^2 - r^2)} = \frac{\pi}{2}\left[1 - \left\{\frac{t^2 - a^2}{t^2 - s^2}\right\}^{1/2}\right]; \qquad s < a. \tag{A3}$$

Hence

$$I = -s\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2 - s^2)^{3/2}}. \tag{A4}$$

Substituting the value of $f_2(t)$ given by (26) in (A4) we have

$$I = -\frac{2s}{\pi}\int_b^\infty \frac{t\,\mathrm{d}t}{(t^2 - s^2)^{3/2}}\int_b^t \frac{F_2(u)\,\mathrm{d}u}{(t^2 - u^2)^{1/2}}. \tag{A5}$$

[redacted]

$$I = -\frac{2s}{\pi}\int_b^\infty F_2(u)\,\mathrm{d}u\int_u^\infty \frac{t\,\mathrm{d}t}{(t^2 - u^2)^{1/2}(t^2 - s^2)^{3/2}}. \tag{A6}$$